PHYS 1220 Exam 4

Brief Solutions

1. Maxwell's equations

Gauss's law for electric charge: An electric charge creates a field pointing away from itself if the charge is positive, or toward itself if the charge is negative.

Gauss's law for magnetic monopoles: Magnetic monopoles do not exist.

Ampère's law: Electric currents and changing electric flux create circulating magnetic fields.

Faraday's law: A changing magnetic field creates a circulating magnetic field.

2. Return of the Toroid electromagnet

A. Magnetic field energy density

$$u_B = \frac{B^2}{2\mu_0}$$

B. Total field energy

$$\begin{aligned} U_B &= \int_b^{b+a} u_B dV \\ &= \int_b^{b+a} \frac{B^2}{2\mu_0} 2\pi r a dr \\ &= \frac{\pi a}{\mu_0} \int_b^{b+a} \left(\frac{\mu_0 N I}{2\pi r} \right)^2 r dr \\ &= \frac{\pi a \mu_0^2 N^2 I^2}{\mu_0 4\pi^2} \int_b^{b+a} \frac{1}{r} dr \\ &= \frac{a\mu_0 N^2 I^2}{4\pi} \ln\left(\frac{b+a}{b} \right) \end{aligned}$$

3. Inductor at a constant voltage

A. Plot of current vs. time

The plot should be a straight line starting at zero ($I = 0$ at $t = 0$).

B. Formula for current as a function of time

We have

$$\begin{aligned} -V_s &= -L dI/dt \\ \frac{V_s}{L} dt &= dI \\ I &= \frac{V_s}{L} \int_0^t dt \\ I &= \frac{V_s}{L} t \end{aligned}$$

Sure enough, this is a straight line starting at $I = 0$ at $t = 0$. I put in the double negative so that the current here is positive instead of negative. Strictly speaking, the voltage across the inductor is the negative of the source voltage.

4. Inductor at a sinusoidal voltage

A. Plot of current vs. time

We'll expect the current to be sinusoidal as well, but out of phase with the voltage. The voltage across the inductor is proportional to the rate of change of the current. The cosine function (the voltage) starts positive and initially decreases, which means that the current initially has a positive slope.

B. Formula for current as a function of time

We have

$$\begin{aligned} -V_s \cos(\omega t) &= -L dI/dt \\ \frac{V_s}{L} \cos(\omega t) dt &= dI \\ I &= \frac{V_s}{L} \int_0^t \cos(\omega t) dt \\ I &= \frac{V_s}{L} \frac{1}{\omega} \sin(\omega t) \\ I &= \frac{V_s}{\omega L} \sin(\omega t) \end{aligned}$$

5. LR **Circuit powering up**

A. Initial current

The current starts at zero.

B. Eventual current

As the current settles into its eventual value, it won't be changing. Therefore, the voltage across the inductor will be zero, so the voltage across the resistor will be V_s. In that case, the circuit current will be V_s/R.

C. Half life

To find the current at a particular time, we'll need to know the equation for current as a function of time. It will be a saturating exponential of the form $I = I_0 \left(1 - e^{-t/\tau}\right)$. We know that the saturation current $I_0 = V_s/R$, and we also know that the time constant $\tau = L/R$. There is nothing left to find, so the formula is

$$I = \frac{V_s}{R} \left(1 - e^{-Rt/L}\right)$$

At $t = t_{1/2}$, the current is $V_s/(2R)$. So we can solve the formula for I above.

$$\begin{aligned} I &= \frac{V_s}{R} \left(1 - e^{-Rt/L}\right) \\ \frac{V_s}{2R} &= \frac{V_s}{R} \left(1 - e^{-Rt/L}\right) \end{aligned}$$

$$\frac{1}{2} = 1 - e^{-Rt/L}$$
$$e^{-Rt/L} = \frac{1}{2}$$
$$e^{Rt/L} = 2$$
$$Rt/L = \ln 2$$
$$t_{1/2} = \frac{L}{R}\ln 2$$

So we have $t_{1/2} = \ln 2L/R$, or just $t_{1/2} = \tau \ln 2$.

D. Voltages across the inductor

The voltage across the inductor and resistor in series is just the source voltage, so their voltages must add to V_s. Knowing the current, it will be easy to find the resistor voltage. Subtraction then will tell us the inductor voltage.

At $t = 0$, there is zero current and thus zero voltage across the resistor. The entire source voltage therefore is across the inductor, so $V_L = V_s$.

At $t = t_{1/2}$,$I = 1/2\, V_s/R$, so the voltage across the resistor is $V_s/2$. So, then is $V_L = V_s/2$.

at $t = \infty$, $V_L = 0$.

E. Voltages across the resistor

We have already reasoned through most of this in part D.

At $t = 0$, $V_R = 0$.

At $t = t_{1/2}$, $V_R = V_s/2$.

At $t = \infty$, $V_R = V_s$.

6. Resonant LC Circuit

A. Resonant frequency

The formula for the resonant *angular* frequency is $\omega = 1/\sqrt{LC}$. To convert this to frequency f, use $\omega = 2\pi f$. This gives us

$$\omega^2 = \frac{1}{LC} = \frac{1}{(10^{-4}\,\text{F})(3.60 \times 10^{-4}\,\text{H})} = \frac{1}{3.60 \times 10^{-8}\,\text{s}^2} = 2.78 \times 10^7 (\text{rad/s})^2$$
$$\omega = 5270\,\text{rad/s}$$
$$f = \frac{\omega}{2\pi} = 839\,\text{Hz}$$

B. Maximum current

We know that the circuit resonates at $\omega = 1/\sqrt{LC}$. If we say that the voltage follows $V = V_0 \cos(\omega t)$, the capacitor gives us $V = Q/C$ and the inductor gives us $V = -L dI/dt$. We can use either one of these to find the maximum current.

Using the capacitor:

$$Q = CV$$
$$Q = CV_0 \cos(\omega t)$$
$$I = dQ/dt = CdV/dt$$

$$I = CV_0 \left(-\omega \sin(\omega t)\right)$$
$$I = -\omega C V_0 \sin(\omega t)$$

This is a sinusoid with an amplitude of $\omega C V_0$.

$$\omega C V_0 = \frac{1}{\sqrt{LC}} C V_0 = \sqrt{\frac{C^2}{LC}} V_0 = \sqrt{C/L} V_0$$

Using the inductor:

$$V = -L dI/dt$$
$$V_0 \cos(\omega t) = -L dI/dt$$
$$-\frac{V_0}{L} \cos(\omega t) dt = dI$$
$$I = -\frac{V_0}{L} \int \cos(\omega t) dt$$
$$I = -\frac{V_0}{\omega L} (\sin \omega t)$$

This is a sinusoid with amplitude $V_0/(\omega L)$.

$$\frac{V_0}{\omega L} = \frac{V_0 \sqrt{LC}}{L} = \sqrt{C/L} V_0$$

It worked! We get the same value for I_0, the amplitude of the current, by both approaches. This comes out to

$$I_0 = V_0 \sqrt{C/L} = (5.00\,\mathrm{V}) \sqrt{1/3.60}/\Omega = 2.63\,\mathrm{A}$$

7. Step down transformer

The voltage ratio $V_1/V_2 = 2400/120 = 20$. This means that the windings ratio $N_1/N_2 = 20$ as well, and the current ratio $I_1/I_2 = 1/20$.

A. Primary current

We are given $I_2 = 20\,\mathrm{A}$, so $I_1 = I_2/20 = 1.0\,\mathrm{A}$.

B. Primary windings

We are given $N_2 = 600$, so $N_1 = 20 N_2 = 12,000$.