Worksheet 10: Torque and Angular Momentum Answer Key

1. A car of mass 1500 kg skids forward on dry pavement. Its tires have a radius of 35 cm and the coefficient of kinetic friction between the tires and the road is $\mu = 0.8$. The car’s center of mass is 70 cm above the road.

   a. What is the torque on the car about its center of mass from the force of kinetic friction?

   The lever arm is $r = 0.70 \, \text{m}$ and the force of friction is 
   $\mu$ times the car’s weight, $f = \mu mg$. If the car travels in the 
   $+x$ direction, $f$ is in the $-x$ direction and the torque $\tau$ is in 
   the $-z$ direction. So $f = (0.8)(1500 \, \text{kg})(9.8 \, \text{N/kg}) = 11760 \, \text{N}$ 
   and $\tau = hf = 8232 \, \text{Nm}$.

   b. If the wheels are located 1.4 m in front of and behind the car’s center of mass, what fraction of the car’s weight is supported by its front tires during the skid?

   This is a torque statics problem. The total torque on the car has to be zero, though the total force is $f$ in the $-y$ direction. The free-body diagram to the right shows three 
   vertical forces: the car’s weight $mg$ down, and the normal 
   forces front $N_F$ and rear $N_R$. The three forces that cause torques are friction $f$ with lever arm $h = 0.7 \, \text{m}$ and the 
   normal forces each with lever arm $D = 1.4 \, \text{m}$.

   Newton’s second law for the vertical forces gives us $\Sigma F = N_R + N_F - mg = 0$ and 
   the torques give us $\Sigma \tau = DN_F - DN_R - hf = 0$. (The negative signs in the torque 
   equation are because the torques from $f$ and $N_R$ are in the opposite direction 
   from the torque caused by $N_F$.) Our task is to find $N_F$ and $N_R$. Fortunately, we 
   have two equations for these two unknowns. Starting with the torque equation

   
   \[
   DN_F - DN_R - hf = 0
   \]

   \[
   D(N_F - N_R) = hf
   \]

   We now use the force equation to eliminate $N_R$

   \[
   D[N_F - (mg - N_F)] = \mu hmg
   \]

   \[
   2DN_F - Dmg = \mu hmg
   \]

   \[
   N_F = mg(D + \mu h)/(2D) = mg[1/2 + \mu h/(2D)]
   \]

   \[
   N_F = mg(0.5+[(0.8)(0.7 \, \text{m})/(2.80 \, \text{m})] = mg(0.5+0.2) = 0.7 \, mg
   \]

   \[
   N_R = mg - N_F = mg[1-1/2-\mu h/(2D)] = mg[1/2-\mu h/(2D)] = mg(0.5–0.2) = 0.3 \, mg
   \]

   We don’t have to compute these further because they are in terms of the car’s 
   weight $mg$. So the front supports 70% of the car’s weight and the rear only 30%. 
   This is why the rear tires of a car lose traction sooner than the front tires when 
   braking.
c. What is the torque on each wheel about its axis from the force of kinetic friction?

There are two front tires, each supporting \(N_f/2\), and two rear tires, each supporting \(N_r/2\). Thus the front tires each have a friction force \(f_f = \mu N_f/2\) and the rear tires each \(f_R = \mu N_r/2\). The torques are \(\tau = rf\), where \(r = 0.35\, \text{m}\), the radius of a tire. The weight of the car \(mg = 14700\, \text{N}\). This gives \(\tau_f = rf_f = (0.35\, \text{m})(0.8)(0.7)(14700\, \text{N})/2 = (0.7)(2058\, \text{Nm}) = 1440.6\, \text{Nm}\) and \(\tau_R = rf_R = (0.35\, \text{m})(0.8)(0.3)(14700\, \text{N})/2 = (0.3)(2058\, \text{Nm}) = 617.4\, \text{Nm}\).

2. A **ballistic pendulum** is a venerable device used to indirectly measure cannon muzzle velocities. The cannonball is fired horizontally and is immediately caught and held by a pivoted catcher assembly. The catcher swings upward, allowing the initial cannonball velocity to be deduced.

The ball of mass \(m\) is fired at speed \(v\) and caught by a pivoted catcher of mass \(M\), hanging a distance \(R\) below its frictionless pivot. The catcher has a moment of inertia \(I\) about the pivot. The catcher swings to a maximum angle of \(\theta\) from the vertical, raising its center of mass a height \(h\).

a. Use conservation of angular momentum to find the angular speed \(\omega\) of the assembly immediately after catching the cannonball. (Treat the cannonball as a point particle.)

\[
L_i = L_f \\
Rmv = I_T\omega \\
\omega = \frac{Rmv}{I_T}
\]

where \(I_T\) is the total moment of inertia of the assembly (catcher + cannonball). \(I\) of the cannonball is \(mR^2\), so \(I_T = I + mR^2\).

b. Use conservation of mechanical energy to find the maximum height \(h\) attained by the catcher.

\[
K_i + U_i + W_{nc} = K_f + U_f \\
I_T\omega^2/2 + 0 + 0 = m_Tgh
\]

where \(m_T = M+m\) is the total mass of the assembly. Then

\[
h = \frac{I_T\omega^2}{2m_Tg} = \frac{I_f(Rmv/I_T)^2}{2m_Tg} = \frac{(Rmv)^2}{2I_f m_T g} = \frac{(Rmv)^2}{2(I + mR^2)(M + m)g}
\]

c. What is the maximum angle \(\theta\) attained by the catcher?

\(h = R(1 - \cos \theta)\), so \(\cos \theta = 1 - h/R\), or \(\theta = \arccos(1 - h/R)\)
With $h$ found as in part b, 
\[
\frac{h}{R} = \frac{R(mv)^2}{2I_I, m_r g} = \frac{R(mv)^2}{2(l + mR^2)(M + m)g},
\]
and that can be used to find $\theta$.

d. Let’s try it with some numbers. The cannonball’s mass is 5 kg, its initial speed is 200 m/s, the mass of the catcher is 150 kg, the catcher is 5 m from the pivot, and the moment of inertia of the catcher is 3800 kg\cdot m^2.

i. What is the angular speed of the assembly after the catch?

\[
I_I = I + mR^2 = 3820 \text{ kg}\cdot\text{m}^2 + (5 \text{ kg})(5 \text{ m})^2 = 3820 + 125 \text{ kg}\cdot\text{m}^2 = 3945 \text{ kg}\cdot\text{m}^2,
\]

\[
\omega = \frac{Rmv}{I_I}. \text{ so } \omega = (5 \text{ m})(5 \text{ kg})(200 \text{ m/s})/(3945 \text{ kg}\cdot\text{m}^2) = (5000/3945)/\text{s} = 1.267/\text{s}
\]

\[
= 1.267 \text{ radian/s}
\]

ii. How high above its starting height does the catcher swing?

\[
h = I_I \omega^2/(2mg) = (3945 \text{ kg}\cdot\text{m}^2)(1.267/\text{s})^2/[2(155 \text{ kg})(9.8 \text{ N/kg})] = 6333/3038 \text{ m}
\]

\[
= 2.085 \text{ m}
\]

Any of the other equivalent formulas above should work as well.

iii. What is the maximum angle $\theta$ attained by the catcher?

\[
h/R = (2.085 \text{ m})/(5 \text{ m}) = 0.417
\]

\[
\theta = \arccos(1 - 0.417) = \arccos(0.583) = 0.949 \text{ radians} = 54.4^\circ.
\]